

RAMIFIED EXTENSIONS OF DEGREE p AND THEIR HOPF-GALOIS MODULE STRUCTURE

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ABSTRACT. Cyclic, ramified extensions L/K of degree p of local fields with residue characteristic p are fairly well understood. Unless $\text{char}(K) = 0$ and $L = K(\sqrt[p]{\pi_K})$ for some prime element $\pi_K \in K$, they are defined by an Artin-Schreier equation. Additionally, through the work of Ferton, Aiba, de Smit and Thomas, and others, much is known about their Galois module structure of ideals, the structure of each ideal \mathfrak{P}_L^n as a module over its associated order $\mathfrak{A}_{K[G]}(n) = \{x \in K[G] : x\mathfrak{P}_L^n \subseteq \mathfrak{P}_L^n\}$ where $G = \text{Gal}(L/K)$. This paper extends these results to separable, ramified extensions of degree p that are not Galois.

1. INTRODUCTION

Let K be a complete local field with valuation v_K normalized so that $v_K(K^\times) = \mathbb{Z}$ and residue field κ finite of characteristic $p > 0$. This means that either K is a finite extension of the p -adic numbers \mathbb{Q}_p , or K is the field of Laurent series $\kappa((X))$ with X indeterminate. We are interested in ramified extensions L of degree p over K . Certainly, some of these extensions are generated by a root of a prime element $\pi_K \in K$, namely, $L = K(x)$ with $x^p = \pi_K$. Such extensions are special. In $\text{char}(K) = p$, these are the inseparable extensions. We call them *atypical*, and restrict our attention to *typical* extensions, those that *cannot* be generated by a root of a prime element. For these extensions, we are interested in addressing two classical questions.

The first question concerns the defining polynomial. As is well-known, when a typical extension L/K is Galois, it can be defined by a Artin-Schreier polynomial

$$(1) \quad p(x) = x^p - x - \beta \in K[x],$$

with its ramification number b , defined as in [Ser79], satisfying $p \nmid b$ and $v_K(\beta) = -b$ [MW56]. By adjusting the argument of [MW56], as presented in [FV02, Chapter III §2], we prove that every typical extension can be defined by a polynomial of the form

$$(2) \quad p(x) = x^p - \alpha x - \beta \in K[x],$$

where again, the ramification number for L/K is linked in a transparent manner to the valuations of the coefficients. Elsewhere, namely [Ama71], such extensions are defined in terms of Eisenstein polynomials. The value of defining extensions by (2) is that in addition to a transparent description of ramification, other properties can be easily described. Indeed, this is why they were first of interest for global function fields, where they are used to determine the Hasse-Witt invariant [Sul75, LRCMR12].

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The second question concerns Galois module structure, or rather, for a general typical extension, what must be called Hopf-Galois module structure. We begin by introducing the question in its classical setting, when L/K is Galois with $G = \text{Gal}(L/K)$. Here the search is for an integral version of the Normal Basis Theorem. Based upon results of Noether and Leopoldt, the question asks for conditions under which the ring of integers $\mathfrak{O}_L = \{x \in L : v_L(x) \geq 0\}$ in L is free over its associated order $\mathfrak{A}_{K[G]} = \{y \in K[G] : y\mathfrak{O}_L \subseteq \mathfrak{O}_L\}$ in $K[G]$, the largest \mathfrak{O}_K -order in $K[G]$ that acts on \mathfrak{O}_L . For general extensions, this and variations of this question present very difficult problems, and progress starting in the 1970s has been slow. On the other hand, for one specific class of extensions, cyclic of degree p , progress has been good [Fer73, Aib03, dST07, Mar13, Huy14]. One explanation for this progress is that cyclic ramified extensions of degree p naturally possess a *scaffold*. This is discussed in [BCE14, §4.1], although the definition of scaffold, as presented in [BCE14] in its full generality, may be a challenge to digest. For extensions of degree p however, a very simple sufficiency condition is available: If there is an element $x \in L$ with $p \nmid v_L(x)$ and an element $\Psi \in K[G]$ that “acts like” the derivative d/dx on the K -basis $\{x^i\}_{i=0}^{p-1}$ for L over K , there is a scaffold. As we shall see, “acts like” is exact in $\text{char}(K) = p$, namely $\Psi \cdot x^i = ix^{i-1}$ for $0 \leq i < p$. In $\text{char}(K) = 0$, it is approximate:

$$\Psi \cdot x^i \equiv ix^{i-1} \pmod{x^{i-1}\mathfrak{P}_L^{\mathfrak{T}}},$$

where $\mathfrak{P}_L = \{x \in L : v_L(x) > 0\}$ is the prime ideal and the degree of approximation is captured by the integer $\mathfrak{T} \geq 1$, the *tolerance* of the scaffold.

As explained in [BCE14, §4.1], using the scaffold and assuming a lower bound on absolute ramification $v_L(p) > (p-1)(b+2)$, an \mathfrak{O}_K -basis for the associate order of an ideal \mathfrak{P}_L^h , defined as $\{y \in K[G] : y\mathfrak{P}_L^h \subseteq \mathfrak{P}_L^h\}$, can be explicitly described [BCE14, Theorem 3.1]. Furthermore, letting $0 \leq \bar{b} < p$ be the residue of $b \bmod p$, we can conclude that

- (1) $\mathfrak{P}_L^{\bar{b}}$ is free over its associated order.
- (2) \mathfrak{O}_L is free over its associated order if and only if $\bar{b} \mid p-1$.
- (3) The inverse different $\mathcal{D}_{L/K}^{-1}$ is free over its associated order if and only if $\bar{b} = p-1$.

These statements also follow from [Fer73, Aib03, dST07, Mar13, Huy14]. The purpose of this paper is to extend these cyclic results to typical extensions L/K , including those that are not Galois. This is accomplished by first using [Chi89, §2] to identify the unique K -Hopf algebra \mathcal{H} that acts upon L (making L an \mathcal{H} -Galois extension). This Hopf algebra has one generator. We explicitly describe the action of this generator on the K -basis $\{x^i\}_{i=0}^{p-1}$ for L with x satisfying (2), and observe that this yields scaffold for the \mathcal{H} -action on L . Once a scaffold exists, the main results of [BCE14] apply. In particular, the three statements above hold, with the associate order of an ideal \mathfrak{P}_L^h in \mathcal{H} defined as $\{y \in \mathcal{H} : y\mathfrak{P}_L^h \subseteq \mathfrak{P}_L^h\}$. Other structural results hold as well. But for those, we direct the reader to [BCE14].

Remark 1. The focus of this paper is on a uniformity of approach, based upon a certain defining equation. Thus we don’t discuss Hopf-Galois module structure in the setting of atypical extensions. In $\text{char}(K) = p$, these extensions are inseparable. See [BCE14, §5] and [BEK, §6] for two different Hopf algebras that act upon L/K and a discussion of their resulting Hopf-Galois module structure. In $\text{char}(K) = 0$ with L/K Galois, see [Fer73].

1.1. Summary of notation. Let p be a prime. The field K is either a finite extension of the p -adic numbers (in $\text{char}(K) = 0$), or a field of Laurent series (in $\text{char}(K) = p$).

Following common conventions, we use subscripts to denote field of reference. So v_K is the valuation normalized so that $v_K(K^\times) = \mathbb{Z}$, π_K is a prime element in K (with $v_K(\pi_K) = 1$), $\mathfrak{O}_K = \{x \in K : v_K(x) \geq 0\}$ is the ring of integers in K . It has a unique maximal ideal $\mathfrak{P}_K = \{x \in K : v_K(x) \geq 1\}$. The field L is *typical* if it is a ramified extension of K of degree p that is not generated by a p th root of a prime element π_K .

2. TYPICAL EXTENSIONS & RAMIFICATION

Theorem 2.1. *If L/K is a typical extension, there are positive integers: $ef = d \mid p - 1$, $0 \leq t < e$, $\gcd(t, e) = 1$ and $0 < b + pt/e < v_L(p)/(p - 1)$ with $\gcd(b, p) = 1$, as well as two elements: $\alpha, \beta \in K$ satisfying $v_K(\beta) = -b$, and $\alpha = \pi_K^{ft} \gamma^f \mu \in \mathfrak{O}_K$ for two units $\gamma, \mu \in \mathfrak{O}_K^\times$ with μ representing a coset of order f in the quotient group $\kappa^\times / (\kappa^\times)^f$ (recall $\kappa = \mathfrak{O}_K / \mathfrak{P}_K$), such that $L = K(x)$ with*

$$x^p - \alpha^{(p-1)/d} x = \beta.$$

Conversely, every such equation yields a typical extension with ramification number

$$\ell = b + \frac{pt}{e},$$

and different $\mathfrak{D}_{L/K} = \mathfrak{P}_L^{(\ell+1)(p-1)}$. The Galois closure for L/K is $M = K(x, y)$ where $y^d = \alpha$, with degree of inertia f and ramification index ep . To describe the Galois group, let r be an integer of order d modulo p . Let $\rho = r$ in $\text{char}(K) = p$ and let ρ be the Teichmüller character for r (a primitive d th root of unity in the p -adic integers \mathbb{Z}_p such that $\rho \equiv r \pmod{p}$) when $\text{char}(K) = 0$. Then $\text{Gal}(M/K) = \langle \sigma, \tau : \sigma^p = \tau^d = 1, \tau\sigma\tau^{-1} = \sigma^r \rangle$ with $\tau(y) = \rho y$, $\tau(x) = x$, $\sigma(y) = y$, and $\sigma(x) = x + y + y\Delta$. In $\text{char}(K) = p$, $\Delta = 0$. In $\text{char}(K) = 0$, $\Delta \in M$ satisfies $v_M(\Delta) = v_M(p) - (p - 1)e\ell$. The ramification number of $M/K(y)$ is $e\ell$.

Remark 2. In $\text{char}(K) = p$, $v_L(p) = \infty$ and there is no upper bound on ℓ .

Remark 3. As mentioned in §1, our definition of ramification numbers follows [Ser79, IV]. For separable non-Galois extensions, this is done via the upper numbering for the Galois group of the Galois closure of L/K , as suggested by [Ser79, IV§3 Remark 2] and developed more fully in [Hel91]. We need to state this clearly, as there are two alternative ways that ramification numbers can be assigned values. The simplest way to explained difference is in terms of the graph of the Hasse-Herbrand function, which is a piecewise linear function that is graphed either in $[-1, \infty) \times [-1, \infty)$, as in [Ser79] (see graphs on [AT90, pages 116, 117]), or shifted into $[0, \infty) \times [0, \infty)$, as in [Hel91, page 2274] and [JR06, §3]. Ramification numbers then correspond with vertices. Lower ramification numbers corresponding to first coordinates. Upper ramification numbers correspond to second coordinates. If the Hasse-Herbrand function is graphed in $[-1, \infty) \times [-1, \infty)$, we say that the ramification numbers (lower or upper) have Serre values. If the function is graphed in $[0, \infty) \times [0, \infty)$, they have Artin values. To transition from first to the second, simply add 1 to both coordinates. Therefore, when comparing the statement in this theorem with other results, it is important to bear this in mind. We use Serre values. The equation $x^p - \pi_K x = \pi_K$ is used in [Lub13, §1.4 Example 2] with $K = \mathbb{Q}_p$ and $\pi_K = p$ where the ramification number is reported as $p/(p - 1)$. This is an Artin value. Using Theorem 2.1 with $f = t = 1$, $e = d = p - 1$, $b = -1$ and $\alpha = \beta = \pi_K$, the Serre value is $1/(p - 1)$. The difference between the two values is 1.

The rest of this section is concerned with the proof of Theorem 2.1. We begin with an exercise in group theory. Since the residue field κ is finite, the group $G = \text{Gal}(M/K)$ for the Galois closure M/K of L/K is solvable [Ser79, IV §2 Corollary 5]. Any solvable transitive subgroup G of the symmetric group S_p on p letters contains a unique subgroup $\langle \sigma \rangle$ of order p and is contained in the normalizer of $\langle \sigma \rangle$ in S_p (e.g. [DF04, pg. 638 Exercise 20]). Note that $\text{Gal}(M/K)/\langle \sigma \rangle$ is cyclic of order d for some $d \mid (p-1)$. Let M' be the fixed field of $\langle \sigma \rangle$, a cyclic extension of K of degree d . Let $\langle \tau \rangle$ be the subgroup that fixes L . From this it follows that there is an integer r of order d modulo p such that G is as in the statement above. At this point, the elements $\sigma, \tau \in G$ along with the integers d, r are fixed. We characterize M' .

Since the residue field κ contains \mathbb{F}_p^\times , K contains the d th roots of unity. Thus M'/K is Kummer and $M' = K(y)$ with $y^d = \alpha$ for some $\alpha \in K$ representing a coset of order d in the quotient group $K^\times/(K^\times)^d$, and $\tau(y) = \rho y$. Within M' there is a maximally unramified extension of K , which we call K' . Let $e = [M' : K']$ and $f = [K' : K]$. Thus $d = ef$. Let $\pi_K, \pi_{K'}$ denote prime elements in K, K' , respectively. We can replace y by $y\pi_K^i$ and still have a Kummer generator for M'/K , and so we can assume that $0 < v_{M'}(y) \leq e$. Since M'/K' is totally ramified and tame (including the case $e = 1$ where $M' = K'$), $M' = K'(z)$ where $z^e = \pi_{K'}$. Since K'/K is unramified, $\pi_{K'} = \pi_K u$ for some $u \in \mathfrak{O}_{K'}^\times$. Since both y and z are Kummer generators of M'/K' we have $y = z^t \omega$ for some $1 \leq t \leq e$ satisfying $\gcd(t, e) = 1$, and $\omega \in \mathfrak{O}_{K'}^\times$. Thus $y^e = \pi_K^t u^t \omega^e$. Let $\omega' = u^t \omega^e \in \mathfrak{O}_{K'}^\times$. Since K'/K is an unramified Kummer extension, $K' = K(v)$ where $v^f = \mu \in \mathfrak{O}_K^\times$ where μ represents a coset of order f in the quotient group $\kappa^\times/(\kappa^\times)^f$. But $\omega' = y^e/\pi_K^t$ is also a Kummer generator for K' , so $\omega' = v^s \gamma$ for some $1 \leq s < f$ satisfying $\gcd(s, f) = 1$, and $\gamma \in \mathfrak{O}_K^\times$. As a result, $y^d = (y^e)^f = \pi_K^{tf} \gamma^f \mu^s$. But then, without any loss of generality, we can replace μ by μ^s and relabel, since the descriptions of these two elements are the same. Now for the converse, observe that for $y^d = \alpha$ with α as above, $y^e/(\pi_K^t \gamma)$ satisfies the equation $v^f = \mu$ and thus generates an unramified extension of degree f . Furthermore, y satisfies $y^e = \pi_K^t \gamma v \in K'$. Since $\gcd(t, e) = 1$, let $t't \equiv 1 \pmod{e}$. Then $y^{t'}$ satisfies $x^e \in \pi_{K'}(K')^e$ for some prime element $\pi_{K'} \in K'$. In summary, we have found that $M' = K(y)$ where $y^d = \alpha = \pi_K^{tf} \gamma^f \mu^s$ as in the statement of the theorem. Note that $v_{M'}(y) = t$.

Consider the cyclic extension M/M' , which is ramified because L/K is ramified of degree p , and ramification is multiplicative in towers. Assume for a contradiction that M/M' is not typical. So $M = M'(X)$, $X^p = \pi_{M'}$ and $\zeta_p \in M'$ where ζ_p is a primitive p th root of unity, then the norm $z = N_{M/L}(X) = \prod_{i=0}^{d-1} \tau^i(X) \in L$ satisfies $z^p = N_{M/L}(X^p) = \prod_{i=0}^{d-1} \tau^i(\pi_{M'}) \in K$. So $z^p \in K$ where $v_{M'}(z^p) = d$, which means that $v_K(z^p) = f$. Since $\gcd(f, p) = 1$, this means that we can generate L by a p th root of a prime element, contradicting our assumption that L/K is typical. We conclude that M/M' is a typical Galois extension, which means that $M = M'(X)$ where $X^p - X = \beta'$ for some $\beta' \in M'$ with $v_{M'}(\beta') = -b'$, $1 \leq b' < v_M(p)/(p-1)$ with $p \nmid b'$ and $(\sigma - 1)X = 1 + \varepsilon$ where in $\text{char}(K) = p$ we have $\varepsilon = 0$. In $\text{char}(K) = 0$ we have $\varepsilon \in M$ with $v_M(\varepsilon) = v_M(p) - (p-1)b' > 0$ [FV02, Chap. 3, §2].

Let $\mathbb{X} = yX$. Observe that $v_M(\mathbb{X}) = pt - b'$, and set

$$(3) \quad x = \frac{1}{d}(1 + \tau + \cdots + \tau^{d-1})\mathbb{X} \in L.$$

Let G_i be the ramification filtration for $\text{Gal}(M/K)$, then $G_i \cap \langle \sigma \rangle$ yields the ramification filtration for $\text{Gal}(M/L) = \langle \tau \rangle$. As a result the maximal unramified extension of L , called

L' , satisfies $[M : L'] = e$ and $[L' : L] = f$. The different for M/L , $\mathfrak{D}_{M/L}$, is \mathfrak{P}_M^{e-1} . Let $\text{Tr} = 1 + \tau + \cdots + \tau^{d-1}$ denote the trace for M/L . Since Tr is \mathfrak{D}_L -linear, $\text{Tr}(\mathfrak{P}_M^n)$ is an ideal in \mathfrak{D}_L . For any integer r , [Ser79, Ch. 3, Prop. 7] shows that $\text{Tr}(\mathfrak{P}_M^n) \subseteq \mathfrak{P}_L^r$ if and only if $\mathfrak{P}_M^n \subseteq \pi_L^r \mathfrak{D}_{M/L}^{-1} = \mathfrak{P}_M^{er-e+1}$. Thus $r \leq (n + e - 1)/e$, and so $\text{Tr}(\mathfrak{P}_M^n) = \mathfrak{P}_L^r$ where $r = 1 + \lfloor (n - 1)/e \rfloor = \lceil n/e \rceil$. This proves that $v_L(x) = \lceil (pt - b')/e \rceil$. Let us set

$$b = - \left\lceil \frac{pt - b'}{e} \right\rceil.$$

So $b' = eb + pt + r$ for some $0 \leq r < e$, and $v_M(x/y) = -eb - pt = r - b' \geq -b' = v_M(X)$.

In the next step of this argument, we will identify an element $\lambda \in L$ such that $L = K(\lambda)$, $v_L(\lambda) = -b$, $(\sigma - 1)(\lambda/y) \in 1 + \mathfrak{P}_M$, and $\lambda^p - \alpha^{(p-1)/d}\lambda \in K$. In $\text{char}(K) = p$, this is $\lambda = x$. In $\text{char}(K) = 0$, the process is more complicated, and thus the two arguments will diverge. But before they diverge, observe that as soon as we identify an element $\lambda \in L$ such that $(\sigma - 1)(\lambda/y) \in 1 + \mathfrak{P}_M$, we may conclude that $r = 0$. Indeed,

$$b' = eb + pt \text{ and } p \nmid b.$$

The reason is that $v_M((\sigma - 1)\mu) \geq v_M(\mu) + b'$ for all $\mu \in M$, and if $r \neq 0$, then $v_M(\lambda/y) > v_M(X) = -b'$, which would imply $(\sigma - 1)(\lambda/y) \in \mathfrak{P}_M$. Additionally, as soon as we prove $x^p - \alpha^{(p-1)/d}x = \beta$ for some $\beta \in K$, we can observe that since $0 < b' = eb + pt$, we have $-b < pt/e$, which means that $v_L(x^p) < v_L(\alpha^{(p-1)/d}x)$ and thus $v_K(\beta) = -b$. Note that $p \nmid b$ and

$$0 < b + pt/e < pv_K(p)/(p - 1).$$

Determining λ for $\text{char}(K) = p$. Assume $\text{char}(K) = p$. For $1 \leq i < d$, check that $\sigma\tau^i = \tau^i\sigma^{r^{-i}}$. Since $\rho = r$, using (3) we have

$$\begin{aligned} \sigma x &= \frac{\sigma}{d}(1 + \tau + \cdots + \tau^{d-1})\mathbb{X} = \frac{1}{d}(\sigma + \tau\sigma^{r^{-1}} + \tau^2\sigma^{r^{-2}} + \cdots + \tau^{d-1}\sigma^{r^{-(d-1)}})\mathbb{X} \\ &= \frac{1}{d}((\mathbb{X} + y) + \tau(\mathbb{X} + r^{-1}y) + \tau^2(\mathbb{X} + r^{-2}y) + \cdots + \tau^{d-1}(\mathbb{X} + r^{-(d-1)}y)) = x + y, \end{aligned}$$

which means that $(\sigma - 1)x = y$. Therefore $\sigma^i x = x + iy$ for $0 \leq i < d$. Now the norm of x , namely $N_{M/M'}(x) = \prod_{i=0}^{p-1} \sigma^i x$ equals

$$\prod_{i=0}^{p-1} (x + iy) = y^p \prod_{i=0}^{p-1} \left(\frac{x}{y} + i \right) = y^p \left(\frac{x^p}{y^p} - \frac{x}{y} \right) = x^p - \alpha^{(p-1)/d}x.$$

Clearly $x^p - \alpha^{(p-1)/d}x$, as a norm, is fixed by σ , but because $\alpha^{(p-1)/d} \in K$, it is also fixed by τ . As a result, $x^p - \alpha^{(p-1)/d}x \in K$.

Determining λ for $\text{char}(K) = 0$. In this case, x , as defined by (3), only provides us with a first approximation for λ . We will set $\lambda_0 = 0$, $\lambda_1 = x \in L$, and construct a sequence $\{\lambda_n\} \subset L$ satisfying certain properties such that $\lambda = \lim \lambda_n$ gives us the desired element. First, we need three preliminary results. Observe that $v_L(\lambda_1) = v_L(x) = -b$.

Lemma 2.2. $(\sigma - 1)x = y + y\Delta_1 \in M$ where $v_M(\Delta_1) > b' - (eb + pt) \geq 0$.

Proof. Recall that $M = M'(X)$ where X satisfies an Artin-Schreier equation and $(\sigma - 1)X = 1 + \varepsilon$ where $v_M(\varepsilon) = v_M(p) - (p - 1)b'$. Furthermore, $\mathbb{X} = yX$ and thus $\sigma^i \mathbb{X} = \mathbb{X} + iy + (1 + \sigma + \dots + \sigma^{i-1})y\varepsilon \equiv \mathbb{X} + iy + iy\varepsilon \pmod{y\varepsilon \mathfrak{P}_M^{b'}}$ for $0 \leq i < p$. In $\text{char}(K) = 0$, we don't have $\rho = r$, but we do have $\rho \equiv r \pmod{p}$. So for $0 \leq j < d$, given r^{-j} , we may define $\bar{r}_j \equiv r^{-j} \pmod{p}$ with $0 \leq \bar{r}_j < p$. This means that $\tau^j \sigma^{r^{-j}} \mathbb{X} = \tau^j \sigma^{\bar{r}_j} \mathbb{X} \equiv \tau^j \mathbb{X} + \rho^j \bar{r}_j y + \tau^j \bar{r}_j y\varepsilon \pmod{y\varepsilon \mathfrak{P}_M^{b'}}$ where $\rho^j \bar{r}_j \equiv 1 \pmod{p}$. Since $v_M(p) \geq v_M(\varepsilon) + b'$, we find that $\tau^j \sigma^{r^{-j}} \mathbb{X} \equiv \tau^j \mathbb{X} + y + y\tau^j \varepsilon \pmod{y\varepsilon \mathfrak{P}_M^{b'}}$. Therefore

$$\begin{aligned} \sigma x &= \frac{\sigma}{d}(1 + \tau + \dots + \tau^{d-1})\mathbb{X} = \frac{1}{d}(\sigma + \tau\sigma^{r^{-1}} + \tau^2\sigma^{r^{-2}} \dots \tau^{d-1}\sigma^{r^{-(d-1)}})\mathbb{X} \\ &\equiv x + \frac{1}{d} \sum_{j=0}^{d-1} y + \frac{y}{d} \sum_{j=0}^{d-1} \tau^j \varepsilon \equiv x + y + \frac{y}{d} \text{Tr}(\varepsilon) \pmod{y\varepsilon \mathfrak{P}_M^{b'}}, \end{aligned}$$

where Tr is the trace for M/L . Recall that $\text{Tr}(\mathfrak{P}_M^n) = \mathfrak{P}_L^{\lceil n/e \rceil}$. Since $e \mid v_M(p) - (p - 1)b' = v_M(\varepsilon)$ and $v_M(p) > (p - 1)b'$, this means that $v_M(\text{Tr}(\varepsilon)) = v_M(\varepsilon) = v_M(p) - (p - 1)b' \geq e$. We have proven that $(\sigma - 1)x = y + y\Delta_1$ for some $\Delta_1 \in M$ where $v_M(\Delta_1) = v_M(p) - (p - 1)b' \geq e > r = b' - (eb + pt)$. \square

Define

$$\wp_\alpha(X) = y((X/y)^p - X/y) = \frac{1}{y^{p-1}}X^p - X = \frac{1}{\alpha^{(p-1)/d}}X^p - X \in K[X].$$

Lemma 2.3. $v_M((\sigma - 1)\wp_\alpha(x)) > b' - eb$.

Proof. Using Lemma 2.2, we have

$$\begin{aligned} \frac{1}{y}(\sigma - 1)\wp_\alpha(x) &= \left(\frac{x}{y} + 1 + \Delta_1\right)^p - \left(\frac{x}{y} + 1 + \Delta_1\right) - \left(\left(\frac{x}{y}\right)^p - \frac{x}{y}\right) \\ &= \left(\frac{x}{y} + 1 + \Delta_1\right)^p - \left(\frac{x}{y}\right)^p - (1 + \Delta_1) \\ &= \sum_{i=1}^{p-1} \binom{p}{i} \left(\frac{x}{y}\right)^i (1 + \Delta_1)^{p-i} + \sum_{i=1}^{p-1} \binom{p}{i} \Delta_1^i + (\Delta_1^p - \Delta_1). \end{aligned}$$

Multiplying back through by y , it is enough to show that $v_M(py(x/y)^{p-1}) \geq b' - eb$ when $v_M(x/y) \leq 0$, and $v_M(px) \geq b' - eb$ when $v_M(x/y) > 0$, while also showing $v_M(y\Delta_1) \geq b' - eb$. Under $v_M(x/y) \leq 0$, $v_M(py(x/y)^{p-1}) \geq b' - eb$ is equivalent to $v_M(p) \geq b' + (p - 2)(be + pt)$, which follows from $v_M(p) > (p - 1)b'$ and $b' \geq be + pt$. Under $v_M(x/y) > 0$, $v_M(px) \geq b' - eb$ follows from $v_M(p) > (p - 1)b' \geq b'$. This leaves $v_M(y\Delta_1) > b' - eb$, which is equivalent to $v_M(\Delta_1) > b' - (eb + pt) = r$ and follows from Lemma 2.2. \square

We require one more lemma, which is a generalization of [FV02, (2.2) Lemma].

Lemma 2.4. *Given $Y \in L \setminus K$ there is an $y \in K$ such that $v_M((\sigma - 1)Y) = v_M(Y - y) + b'$.*

Proof. Let $\pi_L \in L$ be a prime element, and express $Y = \sum_{i=0}^{p-1} a_i \pi_L^i$ for some $a_i \in K$. For $1 \leq i < p$, $p \nmid v_M(\pi_L^i)$, and thus $v_M((\sigma - 1)\pi_L^i) = v_M(\pi_L^i) + b'$. Let $y = a_0$. Then $v_M((\sigma - 1)Y) = v_M((\sigma - 1)(Y - y)) = v_M((Y - y) + b'$. \square

We are now prepared to follow the argument from [FV02, pg. 76] by constructing a sequence $\{\lambda_n\} \subset L$ that satisfies the following conditions

$$(4) \quad \begin{aligned} v_L(\lambda_n) &= -b, & v_L(\lambda_{n+1} - \lambda_n) &\geq v_L(\lambda_n - \lambda_{n-1}) + 1, \\ v_M((\sigma - 1)\wp_\alpha(\lambda_{n+1})) &\geq v_M((\sigma - 1)\wp_\alpha(\lambda_n)) + 1, \end{aligned}$$

with \wp_α defined in Lemma 2.3. Once we have done this, we will set $\lambda = \lim \lambda_n$, and observe that $v_L(\lambda) = -b$ and $\wp_\alpha(\lambda) \in K$. To do this, we will define an auxiliary sequence

$$\delta_n = (\sigma - 1)\wp_\alpha(\lambda_n).$$

Using Lemma 2.3, $v_M(\delta_1) > b' - eb$, which since $b' \geq eb + pt$, also means that $v_M(\delta_1) > pt = v_M(y)$. If we ever have $\delta_n = 0$ then since $\wp_\alpha(\lambda) \in K$. Simply set $\lambda = \lambda_n$. This means that we can assume throughout the argument we can assume that $\delta_n \neq 0$, and by induction that $v_L(\lambda_n) = -b$, $v_M(\delta_n) > b' - eb \geq pt = v_M(y)$, and $(\sigma - 1)\lambda_n = y + y\Delta_n$ where $v_M(\Delta_n) > b' - (eb + pt)$.

Using Lemma 2.4, there is a $c_n \in K$ such that $v_M(\delta_n) = v_M(\wp_\alpha(\lambda_n) + c_n) + b'$. Put $\mu_n = \wp_\alpha(\lambda_n) + c_n \in L$ and set

$$\lambda_{n+1} = \lambda_n + \mu_n \in L.$$

Record that $v_M(\mu_n) = v_M(\delta_n) - b'$ and that $(\sigma - 1)\mu_n = \delta_n$. Since $v_M(\mu_n) > -eb$, the first statement means that $v_L(\lambda_{n+1}) = -b$. Using the second statement, $(\sigma - 1)\lambda_{n+1} = (\sigma - 1)\lambda_n + \delta_n$. Thus $(\sigma - 1)\lambda_{n+1} = y + y\Delta_{n+1}$ where $y\Delta_{n+1} = y\Delta_n + \delta_n$. Therefore, $v_M(\Delta_{n+1}) > b' - (eb + pt)$. Since $v_L(\lambda_{n+1} - \lambda_n) = v_L(\mu_n)$ and $v_M(\mu_n) = v_M(\delta_n) - b'$, all that remains of (4) to be verified, is that $v_M(\delta_{n+1}) \geq v_M(\delta_n) + 1$, and this is the next result.

Lemma 2.5.

$$\delta_{n+1} = (\sigma - 1)\wp_\alpha(\lambda_{n+1}) = (\sigma - 1)\wp_\alpha(\lambda_n + \mu_n) \equiv 0 \pmod{\delta_n \mathfrak{P}_M}.$$

Proof. Using the definition of δ_n , this is the same as proving that

$$(\sigma - 1)(\wp_\alpha(\lambda_n + \mu_n) - \wp_\alpha(\lambda_n)) = (\sigma - 1)y^{1-p} \sum_{i=1}^{p-1} \binom{p}{i} \lambda_n^i \mu_n^{p-i} + (\sigma - 1)\wp_\alpha(\mu_n) \equiv 0 \pmod{\delta_n \mathfrak{P}_M}.$$

There are two summands to consider. Consider the first. Note that

$$v_M \left((\sigma - 1)y^{1-p} \sum_{i=1}^{p-1} \binom{p}{i} \lambda_n^i \mu_n^{p-i} \right) \geq b' - (p-1)pt + v_M(p) - (p-1)eb + v_M(\mu_n).$$

Since $v_M(p) > (p-1)b' \geq (p-1)(eb + pt)$ and $v_M(\mu_n) = v_M(\delta_n) - b'$, it follows that the first summand is $0 \pmod{\delta_n \mathfrak{P}_M}$. Consider the second. Note that

$$(\sigma - 1)\wp_\alpha(\mu_n) = \wp_\alpha(\mu_n + \delta) - \wp_\alpha(\mu_n) = y^{1-p} \sum_{i=1}^{p-1} \binom{p}{i} \mu_n^i \delta_n^{p-i} + \wp_\alpha(\delta_n).$$

For $1 \leq i \leq p-1$, $v_M(y^{1-p} \binom{p}{i} \mu_n^i \delta_n^{p-i}) = v_M(p) - (p-1)pt + pv_M(\delta_n) - ib' \geq v_M(p) - (p-1)(pt + eb) + (p-1-i)b' + v_M(\delta_n) \geq v_M(p) - (p-1)(pt + eb) + v_M(\delta_n) \geq v_M(p) - (p-1)b' + v_M(\delta_n) > v_M(\delta_n)$. Furthermore, since $v_M(\delta_n/y) > 0$, we also have $\wp_\alpha(\delta_n) \equiv 0 \pmod{\delta_n \mathfrak{P}_M}$. \square

We have proven that there is a $\lambda \in L$ such that $v_L(\lambda) = -b$, $(\sigma - 1)(\lambda/y) \in 1 + \mathfrak{P}_M$, and

$$\lambda^p - \alpha^{(p-1)/d} \lambda \in K.$$

2.1. Ramification. We turn now to the ramification break number for the extensions described in Theorem 2.1. The ramification number for M/M' is $b' = eb + pt$. The Herbrand function for M/K , using numbering as in [Ser79] and graph from [AT90, page 116], contains a segment of slope $1/e$ from the origin to $x = eb + pt$. Then a ray of slope $1/(ep)$. We are interested in the vertex $(eb + pt, b + pt/e)$, as it gives the largest upper ramification number for M/K as $b + pt/e$, and thus gives $b + pt/e$ as the upper ramification number for L/K . Since the Herbrand function for L/K has a segment of slope 1 to $y = b + pt/e$, followed by a ray of slope $1/p$, the upper and lower ramification numbers for L/K agree and equal

$$(5) \quad \ell = b + pt/e.$$

Unless $e = 1$, ℓ is not an integer.

2.2. Different. Using the fact that $\mathfrak{D}_{M/K} = \mathfrak{D}_{M/L}\mathfrak{D}_{L/K}$ [Ser79, III §4 Proposition 8] along with the formula for the exponent on the different in the Galois case, namely [Ser79, IV §Proposition 4], we see that $\mathfrak{D}_{M/K} = \mathfrak{P}_M^{(ep-1)+(eb+pt)(p-1)}$ and $\mathfrak{D}_{M/L} = \mathfrak{P}_M^{e-1}$. Therefore

$$\mathfrak{D}_{L/K} = \mathfrak{P}_L^{(b+1)(p-1)+pt\frac{(p-1)}{e}}.$$

Thus the same expression for the exponent of the different in terms of the ramification number for the extension holds regardless of whether L/K is Galois or not.

3. HOPF-GALOIS MODULE STRUCTURE

Greither and Pareigis have classified the finitely many Hopf-Galois structures that are possible for a given separable extension [GP87]. Childs has showed that there is only one such structure when we restrict to separable extensions L/K of degree p [Chi89, §2], which means that there is only one for typical extensions. While Childs assumes $\text{char}(K) = 0$, his argument applies equally well in $\text{char}(K) = p$. Here, we provide a sketch of [Chi89, §2], relaying on [Chi00] for some of the details. The Hopf algebra \mathcal{H} that provides the unique Hopf-Galois structure is described by descent. When the extension is Galois, \mathcal{H} is just the group algebra $K[G]$. In any case, Theorem 2.1 then allows a simple, explicit description of the action of \mathcal{H} on L/K . Without any adjustment, a scaffold exists for this action. We close with a discussion of what this means for the \mathcal{H} -Galois module structure of the ideals of L .

3.1. Hopf-Galois structure.

Theorem 3.1 (Childs). *Adopt the notation of Theorem 2.1 for a given a typical extension L/K . Recall that $d \mid p-1$. Let $ds = p-1$, and let r_0 denote a primitive root modulo p with Teichmüller representative ρ_0 such that $r_0^s \equiv r \pmod{p}$ and $\rho_0^s = \rho$. Set $\Psi = -1/y \sum_{k=0}^{p-2} \rho_0^{-k} \sigma^{r_0^k}$. Then the unique Hopf algebra \mathcal{H} such that L/K is a \mathcal{H} -Galois extension is explicitly*

$$\mathcal{H} = K[\Psi].$$

It is contained in the group ring $K[y][\langle \sigma \rangle]$ and inherits its counit $\varepsilon(\Psi) = 0$, antipode $S(\Psi) = -\Psi$ and antipode Δ from $K[y][\langle \sigma \rangle]$. For example, in $\text{char}(K) = p$, explicitly

$$\Delta(\Psi) = \Psi \otimes 1 + 1 \otimes \Psi - \alpha^s \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} \Psi^i \otimes \Psi^{p-i}.$$

Proof. As Childs explains in [Chi89, §2], the unique Hopf algebra \mathcal{H} is described by descent. Using our notation, the group algebra $M'[\langle\sigma\rangle]$ where $M' = K(y)$ has K -basis $\{y^i\sigma^j : 0 \leq i < d, 0 \leq j < p\}$. The action of $\langle\tau\rangle$ on these basis elements is given by $\tau^k(y^i\sigma^j) = (\rho^k y)^i \sigma^{jr^k} = (\rho_0^{sk} y)^i \sigma^{jr_0^{sk}}$ with the Hopf algebra $\mathcal{H} = M'[\langle\sigma\rangle]^{\langle\tau\rangle}$ determined to be the subalgebra of $M'[\langle\sigma\rangle]$ fixed by τ . The counit ε , antipode S and comultiplication Δ for \mathcal{H} are determined in $M'[\langle\sigma\rangle]$.

Given a basis element for $M'[\langle\sigma\rangle]$, the sum over its orbit under $\langle\tau\rangle$ certainly lies in $M'[\langle\sigma\rangle]^{\langle\tau\rangle}$. There is only one element in the orbit of $y^0\sigma^0$, namely 1. The orbit of $y^i\sigma^0$ for $i \neq 0$ is a sum of d th roots of unity that equals zero. Otherwise, we are considering the orbit generated by $y^i\sigma^j$ where $j \neq 0$ represents some coset of $\mathbb{F}_p^\times/\langle r \rangle$. A complete set of coset representatives for $\mathbb{F}_p^\times/\langle r \rangle$ is given by $\{r_0^t : 0 \leq t < s\}$. And so we are considering the orbit of $y^i\sigma^{r_0^t}$, namely

$$y^i\theta(i, t) = y^i \sum_{k=0}^{d-1} \rho_0^{isk} \sigma^{r_0^{t+sk}}.$$

These orbits biject with $\{(i, t) : 0 \leq i < d, 0 \leq t < s\}$, a set with $ds = p - 1$ elements. Together with 1, we have K -basis of dimension p for \mathcal{H} .

We would like now, as in [Chi89, §2], to perform a change in basis. First, we introduce, mechanically, the basis change from [Chi89, §2]. Second, we motivate everything based upon [Chi00, §16]. Observe that $\theta(i, t) = \theta(i + bd, t)$ for all $b \in \mathbb{Z}$, and for $0 \leq i < p$, let

$$\Theta(i) = \sum_{t=0}^{s-1} \rho_0^{it} \theta(i, t) = \sum_{k=0}^{p-2} \rho_0^{ik} \sigma^{r_0^k}.$$

The idea is to replace, for a fixed i in $0 \leq i < d$, the s elements $\{y^i\theta(i, t) : 0 \leq t < s\}$ in our basis with the alternate s elements $\{y^i\Theta(i + bd) : 0 \leq b < s\}$. Since $y^{i+bd} = \alpha^b y^i \in K^\times y^i$, this is the same as replacing them with $\{y^{i+bd}\Theta(i + bd) : 0 \leq b < s\}$. Clearly, $\{y^i\Theta(i + bd) : 0 \leq b < s\}$ is contained in the K -span of $\{y^i\theta(i, t) : 0 \leq t < s\}$. Furthermore since $\sum_{b=0}^{s-1} \rho_0^{(t-a)(i+bd)} = s\delta_{t,a}$ where $\delta_{t,a}$ is the Kronecker delta function, we have $sy^i\theta(i, a) = \sum_{b=0}^{s-1} \rho_0^{-a(i+bd)} y^i\Theta(i + bd)$ and thus find that the K -spans are equal. This means that $\{1\} \cup \{y^i\Theta(i) : 0 \leq i \leq p-2\}$ is a K -basis for \mathcal{H} . Since $\sum_{k=0}^{p-2} \rho_0^{ik} = 0$ unless $(p-1) \mid i$, we see that $\{y^i\Theta(i) : 1 \leq i \leq p-2\}$ lies within the augmentation ideal $\mathcal{H}^+ = \{h \in \mathcal{H} : \varepsilon(h) = 0\}$. Furthermore, $\varepsilon(\Theta(0)) = (p-1)$, thus $\Theta(0) - (p-1) \in \mathcal{H}^+$ as well. We now adjust Childs' basis very slightly to one more amenable to our purposes. Set $j = p - i - 1$ and for $1 \leq i < p-1$, set

$$\Psi_j = -\frac{y^i\Theta(i)}{\alpha^s} = \frac{-1}{y^j} \sum_{k \in \mathbb{Z}/(p-1)\mathbb{Z}} \rho_0^{-jk} \sigma^{r_0^k},$$

and additionally, $\Psi_{p-1} = -(\Theta(0) - (p-1))/y^{p-1}$. Thus $\{\Psi_j : 1 \leq j \leq p-1\}$ is a K -basis for \mathcal{H}^+ .

We now use [Chi00, §16] to explain this choice of basis, and find, as a result of this explanation, that $\mathcal{H} = K[\Psi_1]$. In [Chi00, §16], a homomorphism is defined from \mathbb{F}_p^\times to the group of Hopf algebra automorphisms of \mathcal{H} . Let χ be the identity map in $\text{char}(K) = p$, and the Teichmüller character such that the primitive root $r_0 \in \mathbb{F}_p^\times$ maps to $\rho_0 \in \mathbb{Z}_p^\times$ in $\text{char}(K) = 0$. Given $m \in \mathbb{F}^\times$, the automorphism is denoted by $[m]$. It is proven that \mathcal{H}^+ is a $\mathbb{Z}_p[\mathbb{F}_p^\times]$ -submodule of \mathcal{H} in $\text{char}(K) = 0$ or an $\mathbb{F}_p[\mathbb{F}_p^\times]$ -submodule in $\text{char}(K) = p$.

[Chi00, Lemma 16.2]. In either case, the idempotent elements of the group ring decompose $\mathcal{H}^+ \cong \bigoplus_{j=1}^{p-1} \mathcal{H}_j$ into one-dimensional K -spaces $\mathcal{H}_j = \{h \in \mathcal{H}^+ : [m](h) = \chi^j(m)h\}$, an eigenspace decomposition. Since $[m](\sigma) = \sigma^m$, one can check that $\mathcal{H}_j = K\Psi_j$, which explains the significance of the basis that we have chosen. Let $x_i = y^i\Psi_i$ so that x_i agrees with notation in [Chi00, §16]. The argument leading to [Chi00, Proposition 16.5] proves that $K[x_1]$ equals the K -span on $\{1, x_1, \dots, x_{p-1}\}$. This implies $K[\Psi_1] = \mathcal{H}$ as well, and so for the statement in the theorem, set $\Psi = \Psi_1$.

In $\text{char}(K) = p$, it is easy to show that $x_1^i = i!x_i$ for $1 \leq i < p$, and thus this is something we do in Lemma 3.2. As a result, using the formula for comultiplication in [Chi00, (16.7)], the formula for comultiplication $\Delta(\Psi)$ in the statement in the theorem follows. In $\text{char}(K) = 0$, there are units $w_i \in \mathbb{Z}_p$ such that $x_1^i = w_i x_i$ that do not a simple description, and thus we leave the formula for $\Delta(\Psi)$ implicit. \square

Lemma 3.2. *Let $\mathbb{Z}_{(p)}$ be the integers localized at p . Then For $1 \leq i \leq p-1$,*

$$\left(-\sum_{k=1}^{p-1} \frac{1}{k} x^k\right)^i \equiv -i! \sum_{k=1}^{p-1} \frac{1}{k^i} x^k \pmod{(p, x^p - 1)}$$

in the polynomial ring $\mathbb{Z}_{(p)}[x]$.

Proof. Since r_0 is a primitive root modulo p , $\sum_{k=1}^{p-1} r_0^{ek} \equiv 0 \pmod{p}$, for any exponent $1 \leq e \leq p-2$. This means that $\sum_{t=2}^{p-1} \frac{1}{t^e} = \sum_{k=1}^{p-2} r_0^{-ek} \equiv -1 \pmod{p}$. It is easy to see that $\frac{1}{t^i(1-t)} = \frac{1}{1-t} + \sum_{e=1}^i \frac{1}{t^e}$. Thus $\sum_{t=2}^{p-1} \frac{1}{t^i(1-t)} = \sum_{t=2}^{p-1} (\frac{1}{1-t} + \sum_{e=1}^i \frac{1}{t^e}) = \sum_{t=2}^{p-1} (\frac{1}{1-t} + \frac{1}{t}) + \sum_{e=2}^i \sum_{t=2}^{p-1} \frac{1}{t^e} = (\frac{1}{p-1} - 1) + \sum_{e=2}^i -1 \equiv -(i+1) \pmod{p}$. Let $t \equiv k/m \pmod{p}$. This identity becomes $\sum \frac{m^i}{k^i} \frac{m}{m-k} \equiv -(i+1) \pmod{p}$, where the left-hand-sum is over all $1 \leq k \leq p-1$ except $k = m$. This means that $\sum \frac{1}{k^i} \frac{1}{m-k} \equiv \frac{-(i+1)}{m^{i+1}} \pmod{p}$, which allows us to prove by induction that for $1 \leq i \leq p-2$,

$$\left(\sum_{k=1}^{p-1} \frac{1}{k^i} x^k\right) \left(\sum_{k=1}^{p-1} \frac{1}{k} x^k\right) \equiv -(i+1) \sum_{k=1}^{p-1} \frac{1}{k^{i+1}} x^k \pmod{(p, x^p - 1)}.$$

From this the result follows. \square

3.2. Hopf-Galois module structure. Based upon Theorem 3.1, $\mathcal{H} = K[\Psi]$ is the unique Hopf algebra that makes the typical extension L/K Hopf-Galois.

Theorem 3.3. *Let $L = K(x)$ be a typical extension of K , with x as in Theorem 2.1 and ramification number ℓ . Then $\Psi \cdot 1 = 0$ and for $1 \leq i \leq p-1$, $\Psi \cdot x^i \in L$. In particular,*

$$\begin{aligned} \Psi \cdot x^i &= ix^{i-1} && \text{in } \text{char}(K) = p, \\ \Psi \cdot x^i &\equiv ix^{i-1} \pmod{x^{i-1} \mathfrak{P}_L^{v_L(p)-(p-1)\ell}} && \text{in } \text{char}(K) = 0. \end{aligned}$$

Proof. Recall that σ is an automorphism of M/K . Since $\sum_{k=0}^{p-2} \rho_0^{-k} = 0$, $\Psi \cdot 1 = 0$. Because the argument is much simpler for $\text{char}(K) = p$, we treat it first. Note $\sigma^i x = x + iy$ and $\rho_0 = r_0$. Thus $\Psi \cdot x^i = \frac{-1}{y} \sum_{k=0}^{p-2} r_0^{-k} (x + r_0^k y)^i = \frac{-1}{y} \sum_{k=0}^{p-2} \sum_{t=0}^i \binom{i}{t} x^{i-t} r_0^{(t-1)k} y^t = -\sum_{t=0}^i \binom{i}{t} x^{i-t} y^{t-1} \sum_{k=0}^{p-2} r_0^{(t-1)k} = \sum_{t=0}^i \binom{i}{t} x^{i-t} y^{t-1} \delta_{t,1} = ix^{i-1}$, where $\delta_{i,j}$ is the Kronecker delta function. In $\text{char}(K) = 0$, $\sigma x = x + y + y\Delta$ where $\Delta \in M$ with $v_M(\Delta) = v_M(p) - (p -$

$1)(be + pt)$, we need to introduce further notation. Let $1 \leq r_k < p$ satisfy $r_k \equiv r_0^k \pmod{p}$ and set $\Delta_k = (1 + \sigma + \cdots + \sigma^{r_k-1})\Delta$, we have $\sigma^{r_0^k} = \sigma^{r_k} = x + y(r_k + \Delta_k)$ for $1 \leq k \leq p-2$. As a result,

$$\begin{aligned} \Psi \cdot x^i &= \frac{-1}{y} \sum_{k=0}^{p-2} \rho_0^{-k} (x + y(r_k + \Delta_k))^i = \frac{-1}{y} \sum_{k=0}^{p-2} \rho_0^{-k} \sum_{s=0}^i \binom{i}{s} x^{i-s} y^s (r_k + \Delta_k)^s \\ &= \frac{-1}{y} \sum_{s=0}^i \binom{i}{s} x^{i-s} y^s \sum_{k=0}^{p-2} \rho_0^{-k} (r_k + \Delta_k)^s. \end{aligned}$$

Since ρ_0 is a primitive $p-1$ root of unity, $\sum_{k=0}^{p-2} \rho_0^{-k} (r_0^k + \Delta_{r_0^k})^s = 0$ for $s = 0$. Since $\rho_0^{-k} r_k \equiv 1 \pmod{p}$ and $p \equiv 0 \pmod{\Delta}$, we have $\sum_{k=0}^{p-2} \rho_0^{-k} (r_0^k + \Delta_{r_0^k})^s \equiv -\delta_{s,1} \pmod{\Delta}$ for $1 \leq s \leq i$. Since $v_M(y/x) = pt + eb > 0$, we have $x^{i-s} y^s \equiv 0 \pmod{x^{i-1}y}$ for $1 \leq s \leq i$. Therefore $\Psi \cdot x^i \equiv ix^{i-1} \pmod{x^{i-1}\Delta}$. Since $\Psi \in \mathcal{H}$, $\Psi \cdot L \subset L$. The result follows by evaluation $v_L(\Delta)$. \square

The definition of \mathcal{H} -scaffold in [BCE14, Definition 2.3] requires a shift parameter, which is the integer $b_1 = b$ defined in Theorem 2.1, two functions \mathbf{b} and \mathbf{a} , which are $\mathbf{b} : \{0, 1, \dots, p-1\} \rightarrow \mathbb{Z}$ defined by $\mathbf{b}(s) = sb$ and $\mathbf{a} : \mathbb{Z} \rightarrow \{0, 1, \dots, p-1\}$ defined by $\mathbf{a}(t) \equiv -tb^{-1} \pmod{p}$. It requires elements $\lambda_t = x^{\mathbf{a}(t)} \pi_K^{f_t} \in L$ for $t \in \mathbb{Z}$. Let f_t be defined by $t = -\mathbf{a}(t)b + f_t p$. Therefore $v_L(\lambda_t) = t$ and $\lambda_{t_1} \lambda_{t_2}^{-1} \in K$ when $t_1 \equiv t_2 \pmod{p}$, as required. It requires an element $\Psi_1 = \Psi \in \mathcal{H}$ that because Theorem 3.3 satisfies the required properties in order for there to be a \mathcal{H} -scaffold of tolerance

$$\mathfrak{T} = \begin{cases} \infty & \text{in char}(K) = p, \\ v_L(p) - (p-1)\ell & \text{in char}(K) = 0, \end{cases}$$

where, within $\mathbb{Z}_{(p)}$, the integers localized at p , $\ell \equiv b \pmod{p}$.

And so, similar to the discussion in [BCE14, §4.1] we have $0 < \ell < v_L(p)/(p-1)$, and if

$$(6) \quad \ell < \frac{v_L(p)}{p-1} - 2,$$

we can apply [BCE14, Theorem 3.1 and 3.7] to any ideal \mathfrak{P}_L^n to

- (1) determine a basis for its associated order

$$\mathfrak{A}_{\mathcal{H}}(n) = \{h \in \mathcal{H} : h\mathfrak{P}_L^n \subseteq \mathfrak{P}_L^n\},$$

- (2) determine that $\mathfrak{A}_{\mathcal{H}}(n)$ is a local ring, with maximal ideal \mathfrak{M} and residue field $\mathfrak{A}_{\mathcal{H}}(n)/\mathfrak{M} \cong \kappa = \mathfrak{O}_L/\mathfrak{P}_L$,
- (3) determine whether \mathfrak{P}_L^n is free over $\mathfrak{A}_{\mathcal{H}}(n)$. Indeed,
- (4) determine the number of generators for \mathfrak{P}_L^n over $\mathfrak{A}_{\mathcal{H}}(n)$ if it is not free, and
- (5) determine the embedding dimension $\dim_{\kappa}(\mathfrak{M}/\mathfrak{M}^2)$.

Indeed, as a result of the scaffold, the results of [Fer73, Aib03, dST07, Mar13, Huy14] under (6), proven for Galois extensions, hold for non-Galois extensions as well.

In particular, as in the Introduction, if we set $0 < \bar{b} < p$ such that $\bar{b} \equiv b \equiv \ell \pmod{p}$, then

- (1) For all $n \equiv \bar{b} \pmod{p}$, \mathfrak{P}_L^n is free over its associated order $\mathfrak{A}_{\mathcal{H}}(n)$.
- (2) For $n \equiv 0 \pmod{p}$, \mathfrak{P}_L^n is free over $\mathfrak{A}_{\mathcal{H}}(n)$ if and only if $\bar{b} \mid p-1$. This includes \mathfrak{O}_L .

- (3) For $n \equiv \bar{b} + 1 \pmod{p}$, \mathfrak{P}_L^n is free over $\mathfrak{A}_{\mathcal{H}}(n)$ if and only if $\bar{b} = p - 1$. This includes the inverse different $\mathcal{D}_{L/K}^{-1}$.

The first statement follows from [BCE14, Theorem 3.1]. The second and third statements follow from [BCE14, Corollary 3.3].

4. CONCLUDING REMARKS

The definition of a scaffold, as presented in [BCE14], was still evolving when the term, Galois scaffold, was coined in [Eld09]. The intuition, as articulated in [Eld09, §1], was that extensions with Galois scaffolds are somehow extensions that are no more complicated than ramified cyclic extensions of degree p . A more mature intuition is now available and is articulated in [BCE14, §1]. Still the first intuition is useful, and now that scaffolds have been defined more broadly than just for Galois extensions and classical Galois module theory, the question arose whether scaffolds are similarly present in ramified extensions of degree p that are not Galois and their Hopf-Galois structures. This paper considers separable extensions. Elsewhere, evidence is provided for inseparable extensions [BCE14, §5], [BEK, §6].

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REFERENCES

- [Aib03] Akira Aiba, *Artin-Schreier extensions and Galois module structure*, J. Number Theory **102** (2003), no. 1, 118–124. MR 1994476 (2004f:11127)
- [Ama71] Shigeru Amano, *Eisenstein equations of degree p in a \mathfrak{p} -adic field*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **18** (1971), 1–21. MR 0308086 (46 #7201)
- [AT90] Emil Artin and John Tate, *Class field theory*, second ed., Advanced Book Classics, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1990. MR 1043169 (91b:11129)
- [BCE14] Nigel P. Byott, Lindsay N. Childs, and G. Griffith Elder, *Scaffolds and generalized integral Galois modules structure*, Preprint available at arXiv:1308.2088 [math.NT], April, 2014.
- [BEK] Nigel P. Byott, G. Griffith Elder, and Alan Koch, *Action of the λ -divided power Hopf algebra*, Preprint: April 28.
- [Chi89] Lindsay N. Childs, *On the Hopf Galois theory for separable field extensions*, Comm. Algebra **17** (1989), no. 4, 809–825. MR 990979 (90g:12003)
- [Chi00] ———, *Taming wild extensions: Hopf algebras and local Galois module theory*, Mathematical Surveys and Monographs, vol. 80, American Mathematical Society, Providence, RI, 2000.
- [DF04] David S. Dummit and Richard M. Foote, *Abstract algebra*, third ed., John Wiley & Sons, Inc., Hoboken, NJ, 2004. MR 2286236 (2007h:00003)
- [dST07] Bart de Smit and Lara Thomas, *Local Galois module structure in positive characteristic and continued fractions*, Arch. Math. (Basel) **88** (2007), no. 3, 207–219. MR 2305599 (2008b:11120)
- [Eld09] G. Griffith Elder, *Galois scaffolding in one-dimensional elementary abelian extensions*, Proc. Amer. Math. Soc. **137** (2009), no. 4, 1193–1203.
- [Fer73] Marie-Josée Ferton, *Sur les idéaux d’une extension cyclique de degré premier d’un corps local*, C. R. Acad. Sci. Paris Sér. A-B **276** (1973), A1483–A1486. MR 0332733 (48 #11059)

- [FV02] I. B. Fesenko and S. V. Vostokov, *Local fields and their extensions*, second ed., Translations of Mathematical Monographs, vol. 121, American Mathematical Society, Providence, RI, 2002, With a foreword by I. R. Shafarevich. MR 1915966 (2003c:11150)
- [GP87] Cornelius Greither and Bodo Pareigis, *Hopf Galois theory for separable field extensions*, J. Algebra **106** (1987), no. 1, 239–258.
- [Hel91] Charles Helou, *On the ramification breaks*, Comm. Algebra **19** (1991), no. 8, 2267–2279. MR 1123123 (92g:11114)
- [Huy14] Duc Van Huynh, *Artin-Schreier extensions and generalized associated orders*, J. Number Theory **136** (2014), 28–45. MR 3145322
- [JR06] John W. Jones and David P. Roberts, *A database of local fields*, J. Symbolic Comput. **41** (2006), no. 1, 80–97. MR 2194887 (2006k:11230)
- [LRCMR12] Florian Luca, Martha Rzedowski-Calderón, and Myriam Rosalía Maldonado-Ramírez, *A generalization of a lemma of Sullivan*, Comm. Algebra **40** (2012), no. 7, 2301–2308. MR 2948828
- [Lub13] Jonathan Lubin, *Elementary analytic methods in higher ramification theory*, J. Number Theory **133** (2013), no. 3, 983–999. MR 2997782
- [Mar13] Maria Marklove, *Local Galois module structure in characteristic p* , Ph.D. thesis, University of Exeter, 2013.
- [MW56] R. E. MacKenzie and G. Whaples, *Artin-Schreier equations in characteristic zero*, Amer. J. Math. **78** (1956), 473–485. MR 0090584 (19,834c)
- [Ser79] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979, Translated from the French by Marvin Jay Greenberg. MR 554237 (82e:12016)
- [Sul75] Francis J. Sullivan, *p -torsion in the class group of curves with too many automorphisms*, Arch. Math. (Basel) **26** (1975), 253–261. MR 0393035 (52 #13846)

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